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A strong boundedness theorem for dilators

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Abstract

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We prove a strong boundedness theorem for dilators: if $A \subseteq DIL$ is Σ_1^1, then there is a recursive dilator D_0 such that $\forall D \in A$ (D can be embedded into D_0).

1. Introduction

We refer to [1], [2] for background on dilator theory.

The weak or pointwise boundedness theorem for dilators (see [3]) asserts that if $A \subseteq DIL$ (= set of reals coding in some canonical way the countable dilators) is Σ_1^1, then there is a recursive dilator D_0 such that for all $D \in A$, $D(\alpha) \leqslant D_0(\alpha)$, $\forall \alpha$. The purpose of this paper is to prove a strong boundedness theorem where the above conclusion is strengthened to: there is a recursive dilator D_0 such that $\forall D \in A$ (D can be embedded into D_0). Recall that for two dilators D_1 and D_2, D_1 can be embedded into D_2 means that there is a natural transformation $T: D_1 \to D_2$.

2. Statement of the theorem

We denote below by DIL the set of reals coding in some canonical way the countable dilators. For example one can use the formalism of [2] to represent countable dilators as structures in a certain language which are in fact all substructures of a fixed recursively presented countable structure $\mathcal{Q}_1$. These can

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be coded by reals in a straightforward fashion. Abusing notation, if $D \in DIL$ we will use D ambiguously as the real and dilator it codes. We call a dilator *recursive* if it has a recursive code.

If D_1, D_2 are dilators let $D_1 \hookrightarrow D_2$ mean that there is a natural transformation $T: D_1 \to D_2$. If one represents D_1, D_2 as structures (in the formalism of [2]), then $D_1 \hookrightarrow D_2$ holds exactly when D_1 can be embedded into D_2 in the usual model-theoretic sense. Thus we will say that D_1 can be embedded in D_2 if $D_1 \hookrightarrow D_2$.

We can now state our theorem.

Theorem 1. *Let DIL be the set of reals coding (in some canonical way) the countable dilators. If $A \subseteq DIL$, $A \in \Sigma_1^1$, then there is a recursive dilator D_0 such that for all $D \in A$, D can be embedded in D_0.*

3. Proof

We may as well assume that DIL consists of the reals that code subsets of $\mathscr{Q}_1$ (see [2]) which are dilators (any subset of $\mathscr{Q}_1$ is a predilator). Since $A \in \Sigma_1^1$, there is a recursive function $D: \omega^{<\omega} \to$ Finite subsets of $\mathscr{Q}_1$, such that $s \subseteq t \Rightarrow D(s) \subseteq D(t)$ and if $D(\alpha) = \bigcup_n D(\alpha/n)$, for $\alpha \in \mathbb{R} = \omega^\omega$ (=the reals), then $A = \{D(\alpha): \alpha \in \omega^\omega\}$.

3.1. Step 1

We will define first a recursive functor (preserving as usual direct limits and pullbacks) from well-orderings to well-founded relations into which we can embed all $D \in A$. We will call it E.

Definition of E. Given a well-ordering W, the domain of $E(W)$ consists of all pairs (s, x) with $x \in D(s)(W)$,with (s, x) identified with (s', x) if $s \subseteq s'$ and $x \in D(s)(W)$. The relation $<_{E(W)}$ on $E(W)$ is defined by:

$$(s, x) <_{E(W)} (t, y) \Leftrightarrow s, t \text{ are comparable and if } u = s \cup t \text{ then } x <_{D(u)W} y.$$

(Recall that $s \subseteq t \Rightarrow D(s)(W) \subseteq D(t)(W)$.)

Claim. *$<_{E(W)}$ is well-founded.*

Proof of Claim. Assume not and say $\{(s_i, x_i)\}$ is an infinite descending chain. We claim that there is a sequence $n_0 < n_1 < n_2 < \cdots$ such that $s_{n_0} \subseteq s_{n_1} \subseteq s_{n_2} \subseteq \cdots$. Indeed, let n_0 be such that s_{n_0} has minimum length among all s_n. By induction on $n \geq n_0$ we see that $\forall n \geq n_0$ $(s_{n_0} \subseteq s_n)$. Then chose $n_1 > n_0$ so that s_{n_1} has minimum length among all s_n with $n > n_0$, etc.

It is enough now to show that if

$$(t_0, y_0) >_{E(W)} (t_1, y_1) >_{E(W)} (t_2, y_2) >_{E(W)} \cdots >_{E(W)} (t_n, y_n)$$

and $t_0 \subseteq t_i$ for all $i \leqslant n$, then $y_0 >_{D(t_n)(W)} y_n$. Because if we then let $\alpha = \bigcup_i s_{n_i}$, so that $\alpha \in \omega^{<\omega} \cup R$, we have for each i,

$$s_{n_i} >_{D(s_{n_{i+1}})(W)} s_{n_{i+1}}$$

and since $D(s_{n_i})(W) \subseteq D(\alpha)(W)$ for all W, we have

$$s_{n_i} >_{D(\alpha)(W)} s_{n_{i+1}}$$

i.e., $D(\alpha)(W)$ is not well-founded, a contradiction.

We prove now the above fact by induction on $n \geqslant 1$: For $n = 1$ this is immediate by the definition of $<_{E(W)}$. Assume it now true for n. Say

$$(t_0, y_0) >_{E(W)} \cdots >_{E(W)} (t_n, y_n) >_{E(W)} (t_{n+1}, y_{n+1})$$

with $t_0 \subseteq t_i$, $\forall i \leqslant n+1$. Then by induction hypothesis $y_0 >_{D(t_n)(W)} y_n$ and by definition $y_n >_{D(t_n \cup t_{n+1})(W)} y_{n+1}$, $y_0 >_{D(t_n \cup t_{n+1})(W)} y_{n+1}$. But $t_0 \subseteq t_{n+1}$ so y_0, $y_{n+1} \in D(t_{n+1}(W))$ thus $y_0 >_{D(t_{n+1})(W)} y_{n+1}$, and we are done. □

It is easy now to check the following facts:

- $W \subseteq V \Rightarrow (E(W) \subseteq E(V))\ \&\ (<_{E(W)} \subseteq <_{E(V)})$.
- If $W = \bigcup_i W_i$ is a directed union, then

$$E(W) = \bigcup_i E(W_i) \quad \text{and} \quad <_{E(W)} = \bigcup_i <_{E(W_i)}.$$

- If $W, V \subseteq U$, then $E(W) \cap E(V) = E(W \cap V)$. This is because if $(s, x) \in E(W) \cap E(V)$ then $x \in D(s)(W) \cap D(s)(V)$ and $x \in D(s)(W \cap V)$ and $x \in E(W \cap V)$.

Next we will find a natural transformation from each $D(\alpha)$, $\alpha \in \omega^\omega$, into E. For that given any well-ordering W consider $D(\alpha)(W)$. If $x \in D(\alpha)(W)$, let n be the least with $x \in D(\alpha/n)(W)$. Then $x \mapsto (\alpha/n, x)$ is our embedding from $D(\alpha)(W)$ into $E(W)$. If $x, y \in D(\alpha)(W)$ and $x <_{D(\alpha)(W)} y$ and $x \mapsto (\alpha/n, x)$, $y \mapsto (\alpha/m, y)$ are the images of x, y, then since $x <_{D(\alpha/l)(W)} y$ where $l = \max\{m, n\}$, we have $(\alpha/n, x) <_{E(W)} (\alpha/m, y)$. So this is order preserving. Finally we have to show that if $W \subseteq V$ then if $x \in D(\alpha)(W)$, the least n_1 such that $x \in D(\alpha/n_1)(V)$ is the same as the least n_0 such that $x \in D(\alpha/n_0)(W)$. This shows that the embedding $D(\alpha)(W) \to E(W)$ that we defined is a natural transformation, i.e., we have the commutative diagram

$$\begin{array}{ccc} D(\alpha)(W) & \subseteq & D(\alpha)(V) \\ \downarrow & & \downarrow \\ E(W) & \subseteq & E(V) \end{array}$$

Since $D(\alpha/n)(W) \subseteq D(\alpha/n)(V)$ for all n, clearly $n_1 \leqslant n_0$. But if $x \in D(\alpha/n_1)(V)$ then $x =$ "some term t_1 in the trace of $D(\alpha/n_1)$ evaluated at a tuple $\boldsymbol{v}$

from V". But also $x \in D(\alpha/n_0)(W)$; thus also $x =$ "some term t_0 in the trace of $D(\alpha/n_0)$ applied to a tuple $\boldsymbol{w}$ in W". Since $D(\alpha/n_1) \subseteq D(\alpha/n_0)$, $W \subseteq V$ and $D(\alpha/n_0)$ is a dilator, we must have that $t_0 = t_1$ and $\boldsymbol{v} = \boldsymbol{w}$, i.e., $x \in D(\alpha/n_1)(W)$, so that $n_0 \leqslant n_1$ and we are done.

3.2. Step 2

We will now extend each $E(W)$ to a well-ordering $E_0(W)$ (towards building our final dilator D_0):

Let $u_0, u_1, u_2, \ldots$ be a recursive 1–1 enumeration of $\omega^{<\omega}$ so that $u_i \subsetneq u_j \Rightarrow i < j$. Then $<_{E(W)} = \bigcup_n <_n(W)$, where

$$<_n(W) = \{u_n\} \times D(u_n)(W)$$

is of course a well-ordering, and we identify as usual (s, x) with (t, x) if $s \subseteq t$ and $x \in D(s)(W)$. Thus essentially (modulo these identifications) $<_{E(W)}$ is decomposed into a sequence of well-orderings. We will define now a well-ordering $<_n^*(W)$ on the union of the domains of $<_m(W)$ for $m \leqslant n$ such that $<_n^*(W) \supseteq \bigcup_{m \leqslant n} <_m(W)$ and $<_n^*(W) \subseteq <_{n+1}^*(W)$:

Start with $<_n^*(W) = <_0(W)$. Assume $<_n^*(W)$ has been defined. We define $<_{n+1}^*(W)$ by making it agree with $<_n^*(W)$ and $<_{n+1}(W)$ on their respective domains, and for $(s, x) \in \mathrm{dom}(<_n^*(W))$, $(t, y) \in \mathrm{dom}(<_{n+1}(W)) - \mathrm{dom}(<_n^*(W))$, putting:

$$(t, y) <_{n+1}^* (s, x) \Leftrightarrow \exists (u, z) \leqslant_n^* (s, x)(u \subseteq t \;\&\; (t, y) <_{E(W)} (u, z)),$$

and

$$(s, x) <_{n+1}^* (t, y)$$

otherwise.

Let now $E_0(W) = \bigcup_n <_n^*(W)$, so that $E_0(W)$ is a linear ordering with domain $E(W)$, and $<_{E(W)} \subseteq E_0(W)$. We claim that it is actually well-founded, i.e., a well-ordering:

For $\alpha \in E(W)$, let

$$|\alpha| = \min\{n \colon \alpha \in \mathrm{dom}(<_n^*(W))\}.$$

Then if $\alpha_0, \alpha_1, \ldots$ is an infinite descending chain in $E_0(W)$, (towards a contradiction) we can of course assume by taking subsequences, that $|\alpha_0| < |\alpha_1| < \cdots$. Define now inductively,

- $n_0 = \min\{|\alpha_0| \colon$ there is an infinite descending chain $\alpha_0, \alpha_1, \ldots$ in $E_0(W)$ with $|\alpha_0| < |\alpha_1| < \cdots\}$,
- $\bar{\alpha}_0 =$ the $<_{n_0}^*(W)$-least α_0 for which there is an infinite descending chain $\alpha_0, \alpha_1, \ldots$ with $n_0 = |\alpha_0| < |\alpha_1| < \cdots$ in $E_0(W)$,
- $n_1 = \min\{|\alpha_1| \colon$ there is an infinite descending chain $\bar{\alpha}_0, \alpha_1, \ldots$ in $E_0(W)$ with $n_0 < |\alpha_1| < |\alpha_2| < \cdots\}$,
- $\bar{\alpha}_1 =$ etc. . .

We claim now that $\bar{\alpha}_0 >_{E(W)} \bar{\alpha}_1 >_{E(W)} \cdots$ which is a contradiction. Consider for instance $\bar{\alpha}_0 >_{E(W)} \bar{\alpha}_1$. Since $|\bar{\alpha}_1| = n_1 > n_0$ we have that

$$\bar{\alpha}_1 \in \mathrm{dom}(<^*_{n_1}(W)) - \bigcup_{m<n_1} \mathrm{dom}(<^*_m(W)) \quad \text{and} \quad \bar{\alpha}_0 \in \mathrm{dom}(<^*_{n_0}(W)).$$

Since $\bar{\alpha}_0 >_{E(W)} \bar{\alpha}_1$ we must have $\bar{\alpha}_1 <^*_{n_1} \bar{\alpha}_0$ (we write $<^*_n = <^*_n(W)$ from now on), so there is $\alpha' \leqslant^*_{n_1-1} \bar{\alpha}_0$ with $\bar{\alpha}_1 <_{E(W)} \alpha'$. If $\alpha' <^*_{n_1-1} \bar{\alpha}_0$, then since $\alpha', \bar{\alpha}_1, \bar{\alpha}_2, \ldots$ is an $E_0(W)$ infinite descending chain with $|\alpha'| \leqslant n-1 < |\bar{\alpha}_1| < \cdots$, we must have $n_0 = |\bar{\alpha}_0| \leqslant |\alpha'|$. If $|\alpha'| = n_0$, then $\alpha' <^*_{n_0} \bar{\alpha}_0$ (since $<^*_{n_1-1} \supseteq <^*_{n_0}$), and we have contradicted the minimality of $\bar{\alpha}_0$. So $n_1 - 1 \geqslant |\alpha'| > n_0$. Then $\bar{\alpha}_0, \alpha', \bar{\alpha}_1, \ldots$ is an $E_0(W)$ infinite descending chain with $|\bar{\alpha}_0| < |\alpha'| < |\bar{\alpha}_1| < \cdots$ and thus we have contradicted the minimality of n_1. So $\alpha' = \bar{\alpha}_0$ and thus $\bar{\alpha}_1 <_{E(W)} \bar{\alpha}_0$.

3.3. *Step 3*

Finally, to make sure that we have a dilator we modify E_0 a little and put

$$D_0(W) = E_0(\omega \cdot (1+W)).$$

To check that it is a recursive dilator we first check inductively that each

$$D^*_n(W) = <^*_n(\omega \cdot (1+W))$$

is a dilator. Since $D_0(W) = \bigcup_n D^*_n(W)$, so is then D_0. Moreover, each D^*_n is uniformly (on n) recrusive, since the search in the definition of $<^*_{n+1}$ is actually finite. Thus D_0 is also recursive and our proof is complete.

4. Some remarks

The theorem easily extends to n-ptykes, i.e., if PT^n is the set of reals coding in some canonical way the countable n-ptykes, then if $A \subseteq PT^n$ is Σ^1_1 there is a recursive $\Phi_0 \in PT^n$ with $\Phi \hookrightarrow \Phi_0$, $\forall \Phi \in A$.

Also by a similar argument, if $A \subseteq DIL$ is Σ^1_2, then there is a constructible dilator D_0 of cardinality $\aleph_1$ (in the universe V, not in L) such that all $D \in A$ can be embedded in D_0.

For further developments on the subject of embeddability of dilators and ptykes, the reader should consult the paper of Normann and Girard [4]. One of the theorems proved there for example provides a common extension of the result of this paper and the Main Theorem of [3] (see Theorem III.1 of [4]).

References

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